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THE RADON TRANSFORM ON $SL(2,\mathbf{R})/SO(2,\mathbf{R})$

D. I. WALLACE AND RYUJI YAMAGUCHI

ABSTRACT. Let G be $SL(2,\mathbf{R})$. G acts on the upper half-plane $\mathcal{H}$ by the Möbius transformation, providing $\mathcal{H}$ with the Riemannian metric structure along with the Laplacian, Δ. We study the integral transform along each geodesic. G acts on $\mathcal{P}$, the space of all geodesics, in a natural way, providing $\mathcal{P}$ with its invariant measure and its own Laplacian. ($\mathcal{P}$ actually is a coset space of G.) Therefore the above transform can be viewed as a map from a suitable function space on $\mathcal{H}$ to a suitable function space on $\mathcal{P}$. We prove a number of properties of this transform, including the intertwining properties with its Laplacians and its relation to the Fourier transforms.

1. Introduction. Both the Radon transform and the X-ray transform on a symmetric space arise from the problem of reconstructing a function from its integrals along certain paths. For the Euclidean case, a problem can be stated in one of two ways. Suppose all the integrals of some function f along all straight lines are known. It is then possible to reconstruct f from its line integrals. For $\mathbf{R}^2$, the solution to this problem was the inversion of the original Radon-John transform. The ability to invert this particular transform rested in a duality in integral geometry between the points in $\mathbf{R}^2$ and the set of lines in $\mathbf{R}^2$. Technically, the inversion depends heavily on Fourier analysis on $\mathbf{R}^2$. Alternatively, if the space in question is $\mathbf{R}^n$, one might wish to reconstruct f from its integrals over hyperplanes of dimension k. Solutions to both of these problems can be found in Helgason [**4**, **9**]. For the Euclidean case, we shall speak of the Radon transform when we mean an integral over a k-plane where $k < n$, and of the X-ray transform when we mean an integral over a straight line. Some references for the Euclidean case include Helgason [**5**, **9**], Strichartz [**17**], Radon [**13**], Peters [**12**], Solmon [**15**, **16**], and Smith, Solmon and Wagner [**14**].

When one wishes to define the analogous transform on a general symmetric space, several options become available. The source of these options lies in the generalization of the notion of hyperplane. In $\mathbf{R}^2$, for example, there are two ways of thinking of an $(n-1)$-dimensional hyperplane. One is as a totally geodesic $n-1$ submanifold. The term "totally geodesic" means that a straight line tangent to a particular plane is actually contained in it. Thus, one generalization of the Radon transform which has been exploited is the non-Euclidean case of integrals over totally geodesic submanifolds of dimension k. A theory, including inversion formula, has been worked out for the case where k is even, in Helgason [**9**]. More can be found in Gel′fand et al. [**2**], and Lax and Phillips [**11**].

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A second and equally productive way of thinking of an $n-1$ hyperplane in $\mathbf{R}^2$ is as a set of trajectories through a fixed point orthogonal to a family of parallel geodesics. In this case, the non-Euclidean analog of a hyperplane is not a geodesic at all. In fact, for a symmetric space G/K, it can be characterized as the orbit gNg^{-1}, where g is a fixed element of the group and N is the nilpotent part of the Iwasawa decomposition $G = ANK$. Details can be found in Helgason [**10**] or Terras [**18**]. The virtue of this point of view is that the Fourier transform decomposes into two integral transforms. The integral over the surface where the exponential function defined in [**9**] is constant is exactly the above-mentioned orbital integral. This makes the inversion formulas somewhat easier and allows one to explain somewhat the structure of some differential operators on G/K.

In this paper we attempt to construct the results analogous to those for $\mathbf{R}^2$ in the case of symmetric space $\mathcal{H} = SL(2,\mathbf{R})/SO(2,\mathbf{R})$, i.e., the hyperbolic upper half of the complex plane $\mathbf{C}$. The Radon transform is defined, decompositions which mirror those for $\mathbf{R}^2$ are given. The space of geodesics is characterized and the intertwining operators are given for the Laplace-Beltrami operator on $\mathcal{H}$.

2. The space of geodesics $\mathcal{P}$. Let G be $SL(2,\mathbf{R})$, the group of 2×2 matrices with real entries having determinant 1. Let ANK be its Iwasawa decomposition. $A(t)$, $N(n)$ and $R(\theta)$ denote matrices

$$\begin{pmatrix} e^{+t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

respectively. G acts on the complex plane $\mathbf{C}$ by Möbius transformation; let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $SL(2,\mathbf{R})$. It maps an upper half-plane $\mathcal{H} = \{x+iy\colon y>0\}$ into itself. $A(t)$ acts on z as a dilation; $A(t)z = e^t z$. $N(n)$ translates z along the x-axis; $N(n)z = x+n+iy$. $R(\theta)$ rotates z along the circle centered at $(\cosh r)i$ with radius $\sinh r$ for some r. This r is related to the Poincaré metric that is defined below. As it turns out, r is the non-Euclidean distance from the point i to the point z.

A G-invariant measure, in terms of x and y coordinates, is $dxdy/y^2$. We sometimes refer to this measure as dz. The G-invariant differential operator on $\mathcal{H}$ is the Laplacian $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$.

The G-invariant metric on $\mathcal{H}$, called the Poincaré metric, is defined as follows; let u and v be tangent vectors at a point $z = x+iy$, then $\langle u,v\rangle_z = u\cdot v/y^2$. The Riemannian structure endows $\mathcal{H}$ with the set of geodesics $\mathcal{P}$. The set consists of all semicircles with centers on the x-axis and all straight lines parallel to the y-axis in $\mathcal{H}$. The action of G maps geodesics to geodesics, hence G also acts on $\mathcal{P}$. This metric also provides us with a measure on each smooth curve ξ in $\mathcal{H}$ such that it stays invariant when it is mapped into another curve in $\mathcal{H}$ by an element of G. Denote this measure by $d\sigma_\xi$. If ξ is a semicircle with center s and radius r so that $\xi = \{s + r\cos\alpha + ir\sin\alpha\colon 0<\alpha<\pi\}$, then $d\sigma_\xi = d\alpha/(\sin\alpha)$. There is another way to parametrize $\mathcal{P}$ and ξ in it. Each geodesic ξ is $\{R(\theta)N(n)A(t)i\colon -\infty<t<\infty\}$ for some θ and n. Then $d\sigma_\xi = dt$. $\mathcal{P}$ has a manifold structure inherited from G/A. We see later that G/A is a double covering of $\mathcal{P}$.

Let f be a compactly supported smooth function on $\mathcal{H}$. Define the Radon transform $Tf(\xi)$ for each ξ by the equation $Tf(\xi) = \int_\xi f(z)\,d\sigma_\xi(z)$.

In this section, we find a G-invariant measure and a G-invariant second-order differential operator on $\mathcal{P}$. We then compare those to the G-invariant measure and operator on the space of horocycles which has already been studied by Helgason (see [**5**, **6**, and **9**]).

In §3, we define a dual transform of the Radon transform which maps a function on $\mathcal{P}$ back to a function on $\mathcal{H}$. This definition agrees with Helgason [**5**]. The intertwining properties of these transforms with G-invariant differential operators on $\mathcal{H}$ and $\mathcal{P}$ are proved.

In §4, we discuss an inversion formula.

We work with four different parametrizations of $\mathcal{P}$. Since each geodesic is either a semicircle or a straight line parallel to the y-axis, we can parametrize $\mathcal{P}$ with s and r (or $1/r$), where s is the center of the semicircle and r is the radius. When a geodesic is a straight line, then we let $r = +\infty$ (or $1/r = 0$). We can also consider two endpoints x_1 and x_2 of geodesics for parameters of $\mathcal{P}$. We can always choose $x_1 < x_2$ and allow values $+\infty$ for x_2 and $-\infty$ for x_1. Thus the pair $x_1 = -\infty$ and $x_2 = a$, and the pair $x_1 = a$ and $x_2 = +\infty$ represent the same geodesic $x = a$. From x_1 and x_2, we can also choose θ_1 and θ_2 such that $0 \leq \theta_1 < \theta_2 \leq \pi$ and $\cot\theta_1 = -x_1$ and $\cot\theta_2 = -x_2$. θ_1 and θ_2 are then the angles that the straight lines from i to x_1 and x_2 make with the x-axis. The manifold structure of $\mathcal{P}$ is perhaps best expressed with this parametrization. $\mathcal{P}$ is diffeomorphic to the set $\{(\theta_1,\theta_2)\colon 0 \leq \theta_1 < \theta_2 \leq \pi \text{ and if } \theta_1 = 0 \text{ then } \theta_2 \neq \pi\}$ with boundaries $\theta_1 = 0$ and $\theta_2 = \pi$ identified by the equation $(0,\alpha) = (\alpha,\pi)$ for an angle α, $0 < \alpha < \pi$.

We note that the geodesic $x_1 = -\cot\theta_1$ and $x_2 = -\cot\theta_2$ is a rotation $R(\theta_1)$ of a vertical line $x = n$ for some n. In fact, all geodesics are rotations of straight lines.

The fourth parametrization of $\mathcal{P}$ is derived from the Iwasawa decomposition of G. All geodesics are of the form $\{R(\theta)N(n)A(t)i\colon -\infty < t < \infty\}$ for some θ and n. When $\theta = 0$, the geodesic ξ is a straight line $x = n$. The rotation $R(\theta)$ maps two endpoints of this line to the endpoints of a semicircle. x_1 and x_2 can easily be computed from the Möbius transformation $R(\theta)N(n)A(t)$ applied to the point i and by letting t approach $+\infty$. The result is $x_1 = -\cot\theta$ and $x_2 = (nx_1 + 1)/(x_1 - n)$ if $n < \cot\theta$; x_1 and x_2 are switched if $n > \cot\theta$. Here, θ is considered to be in the interval $[0,\pi]$. Two geodesics $\xi_1(t) = R(\theta_1)N(n_1)A(t)i$ and $\xi_2(t) = R(\theta_2)N(n_2)A(t)i$ are the same if either $\theta_1 = \theta_2$ and $n_1 = n_2$ or $\theta_2 = \theta_1 + \cot^{-1}(-n_1)$ and $n_1 = -n_1$. This shows that the coset space G/A is a double covering of $\mathcal{P}$. $\mathcal{P}$ is in fact G/MA where $M = \{I, -I, R(\pi/2), -R(\pi/2)\}$.

PROPOSITION 2.1. *There is a $SL(2,\mathbf{R})$-invariant measure $d\mu$ on $\mathcal{P}$.*

(1) *In terms of center s and radius r,*

$$d\mu = \frac{ds\,dr}{r^2}.$$

(2) *In terms of endpoints x_1 and x_2,*

$$d\mu = \frac{dx_1 dx_2}{2(x_2 - x_1)^2}.$$

(3) *When $x_1 = \cot\theta_1$ and $x_2 = \cot\theta_2$ then*

$$d\mu = \frac{d\theta_1 d\theta_2}{2\sin^2(\theta_1 - \theta_2)}.$$

(4) *When $\xi(t)$ is written as $R(\theta)N(n)A(t)i$ with $n > 0$ then $d\mu = d\theta\,dn$.*

PROOF. We have already discussed the relations between the above four different parametrizations of $\mathcal{P}$. That the four different expressions express the same measure is an easy consequence of these relations.

It remains to prove that $d\mu$ is G-invariant using any of the above four coordinate systems. We choose x_1, x_2 coordinates. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $SL(2, \mathbf{R})$; then g takes endpoints x_1, x_2 of a semicircle to

$$x_1' = \frac{ax_1 + b}{cx_1 + d} \quad \text{and} \quad x_2' = \frac{ax_2 + b}{cx_2 + d}.$$

It is easy to verify that $dx_1' = (cx_1 + d)^{-2} dx_1$ and $dx_2' = (cx_2 + d)^{-2} dx_2$. Also $(x_2' - x_1')^2 = (x_1 - x_2)^2 (cx_1 + d)^{-2} (cx_2 + d)^{-2}$. Hence $dx_1' dx_2' / (x_2' - x_1')^2 = dx_1 dx_2 / (x_2 - x_1)^2$. This proves the invariance of $d\mu$.

PROPOSITION 2.2. *There is an $SL(2, \mathbf{R})$-invariant differential operator $\Box$ on $\mathcal{P}$.*

(1) *In terms of s and r,*

$$\Box = r^2 \left(\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} \right).$$

(2) *In terms of x_1 and x_2,*

$$\Box = \frac{1}{4} (x_2 - x_1)^2 \frac{\partial^2}{\partial x_1 \partial x_2}.$$

(3) *In terms of $x_1 = \cot \theta_1$ and $x_2 = \cot \theta_2$,*

$$\Box = \sin^2(\theta_1 - \theta_2) \frac{\partial^2}{\partial \theta_1 \partial \theta_2}.$$

(4) *When $\xi(t)$ is written as $R(\theta)N(n)A(t)i$, then*

$$\Box = \frac{1}{1 + n^2} \left(\frac{\partial^2}{\partial \theta^2} + 2 \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial n \partial \theta}.$$

PROOF. A change of variables shows that all the expressions in (1)–(4) represent the same differential operator $\Box$. Now, $\Box$ is G-invariant if $\Box$ is invariant under each action $A(t)$, $N(n)$ and $R(\theta)$. From (2), it is easy to see that $\Box$ is invariant under both a dilation $A(t)$ and a translation $N(n)$. A computation shows that a rotation $R(\theta)$ maps a geodesic $x_1 = \cot \theta_1$ and $x_2 = \cot \theta_2$ to a geodesic $x_1' = \cot(\theta_1 + \theta)$ and $x_2' = \cot(\theta_2 + \theta)$. From (3), we see that $\Box$ stays invariant under $R(\theta)$. This proves the proposition.

3. The dual transform and the intertwining properties. Let f be a compactly supported smooth function on $\mathcal{P}$. The Radon transform maps this function f to a function Tf on $\mathcal{P}$ defined by the equation $Tf(\xi) = \int_\xi f(z)\, d\sigma_\xi(z)$. Let $d\xi$ be the G-invariant measure defined in Proposition 2.1. Let ϕ be a compactly supported smooth function on $\mathcal{P}$. The inner product on $\mathcal{P}$,

$$\langle Tf, \phi \rangle = \int_{\mathcal{P}} Tf(\xi) \phi(\xi)\, d\xi,$$

is well defined. The dual transform T^* is characterized by the property that T^* maps a function ϕ on $\mathcal{P}$ to a function $T^*\phi$ on $\mathcal{H}$ such that the inner product on $\mathcal{H}$,

$$\langle f, T^*\phi\rangle = \int_{\mathcal{H}} f(z) T^*\phi(z)\, dz,$$

has the same value as $\langle Tf, \phi\rangle$. To write down $T^*\phi$, we shall first use x, y coordinates for $\mathcal{H}$ and s, r coordinates for $\mathcal{P}$.

PROPOSITION 3.1. *The dual transform T^* of the Radon transform maps a compactly supported smooth function ϕ on $\mathcal{P}$ to $T^*\phi$ on $\mathcal{H}$, given by the equation*

$$T^*\phi(x,y) = \int_{\theta=0}^{\pi} \phi(x - y\cot\theta, y\csc\theta)\, d\theta.$$

PROOF. We have

$$\begin{aligned}\langle Tf, \phi\rangle &= \iint_{\mathcal{P}} Tf(s,r)\phi(s,r)\frac{ds\,dr}{r^2} \\ &= \iint_{\mathcal{P}} \int_{\theta=0}^{\pi} f(s + r\cos\theta, r\sin\theta)\phi(s,r)\frac{d\theta}{\sin\theta}\frac{ds\,dr}{r^2}.\end{aligned}$$

Letting $x = s + r\cos\theta$, $y = r\sin\theta$, and

$$\frac{ds\,dr}{r^2} = \frac{dx\,dy}{\sin\theta}\frac{1}{(y/\sin\theta)^2},$$

we can write the preceding integral as

$$\iint_{\mathcal{H}} \int_{\theta=0}^{\pi} f(x,y)\phi(x - y\cot\theta, y\csc\theta)\, d\theta \frac{dx\,dy}{y^2}.$$

Hence $T^*\phi(x,y) = \int \phi(x - y\cot\theta, y\csc\theta)\, d\theta$.

We now examine the integrand $\phi(x - y\cot\theta, y\csc\theta)$ a little more closely. Here, we are integrating geodesics ξ_θ with respect to θ, where each ξ_θ has a center $s = x - y\cot\theta$ and a radius $y\csc\theta$. We note that $\xi_\theta = \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{smallmatrix}\right)\eta_\theta$, where the matrices are elements of $SL(2,\mathbf{R})$ and act on a geodesic η_θ with center $s = -\cot\theta$ and radius $r = \csc\theta$. η_θ is independent of x and y and passes through the point i in $\mathbf{C}$. So, in fact, we are integrating, with respect to θ, all geodesics ξ_θ which pass through the point $x + iy$. In other words, $T^*\phi(x,y) = \int \phi(s,r)\, d\mu_{x,y}(s,r)$ for a distribution $d\mu_{x,y}$ supported on a subset of $\mathcal{P}$ containing all geodesics having the point $x + iy$ on them. These definitions coincide with those in Helgason [**5**] for a general pair of coset spaces.

In terms of covering G/A, η_θ has coordinates $R(\theta)A$. We can write $T^*\phi$ using the G/K-covering for coordinates of $\mathcal{H}$ and the G/A-covering for coordinates of $\mathcal{P}$:

$$\begin{aligned}T^*\phi&\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}K\right) \\ &= \int_{\theta=0}^{\pi}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}R(\theta)A\right)d\theta\end{aligned}$$

of equivalently

$$T^*\phi(A(t)N(n)K) = \int_{\theta=0}^{\pi} \phi(A(t)N(n)R(\theta)A)\,d\theta.$$

What we would now like to show is the intertwining property of Δ and $\Box$ with respect to T and T^*. This can be expressed as follows:

$$\Delta(T^*\phi) = -T^*(\Box\phi), \qquad \Box(Tf) = -T(\Delta f),$$

where ϕ and f are compactly supported smooth functions on $\mathcal{P}$ and $\mathcal{H}$ respectively.

In order to prove the above, we will rewrite the operators Δ and $\Box$ as a composition of first-order differential operators as follows:

$$\Delta = y^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) = y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial y} + y^2\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right),$$

$$\Box = r^2\left(\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2}\right) = r^2\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\right) - r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + r\frac{\partial}{\partial r}.$$

We write $Dy = y\,\partial/\partial y$, $Dx = \partial/\partial x$, $Dr = r\,\partial/\partial r$ and $Ds = \partial/\partial s$. Each of these operators can be written as an infinitesimal generator of a semigroup operator as follows:

$$Dy\,f = \lim_{p\to 1} \frac{f(x,py) - f(x,y)}{p-1},$$

$$Dx\,f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

etc. This decomposition allows us to bring the generators Dx, Dy, Dr and Ds inside the integral in Lemmas 3.2, 3.3, 3.6 and 3.7. Also, this particular choice of variables allows us to change an action on $\mathcal{H}$ into an action on $\mathcal{P}$.

LEMMA 3.2. $(\partial/\partial x)(T^*\phi) = T^*(\partial\phi/\partial s)$ *for all compactly supported smooth functions* ϕ.

PROOF.

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_{\theta=0}^{\pi} \phi(x - y\cot\theta, y\csc\theta)\,d\theta \\
&\quad= \lim_{h\to 0}\frac{1}{h}\int_{\theta=0}^{\pi} \phi(x+h-y\cot\theta, y\csc\theta) - \phi(x - y\cot\theta, y\csc\theta)\,d\theta \\
&\quad= \int_{\theta=0}^{\pi}\lim_{h\to 0}\frac{\phi(x+h-y\cot\theta, y\csc\theta) - \phi(x - y\cot\theta, y\csc\theta)}{h}\,d\theta \\
&\quad= \int \lim_{h\to 0}\frac{\phi(s+h,r) - \phi(s,r)}{h}\,d\mu_{x,y}(s,r) \\
&\quad= \int \frac{\partial\phi}{\partial s}(s,r)\,d\mu_{x,y}(s,r) \\
&\quad= T^*\left(\frac{\partial\phi}{\partial s}\right)(x,y).
\end{aligned}$$

LEMMA 3.3.

$$y\frac{\partial}{\partial y}(T^*\phi) = T^*\left((s-x)\frac{\partial}{\partial s} + r\frac{\partial\phi}{\partial r}\right)$$

for all compactly supported smooth functions ϕ.

PROOF.

$$
\begin{aligned}
&y\frac{\partial}{\partial y}(T^*\phi(x,y)) \\
&\quad = \lim_{p\to 1}\frac{1}{p-1}\int_{\theta=0}^{\pi}\phi(x-py\cot\theta, py\csc\theta)-\phi(x-y\cot\theta,y\csc\theta)\,d\theta \\
&\quad = \int_{\theta=0}^{\pi}\lim_{p\to 1}\frac{\phi(x-py\cot\theta, py\csc\theta)-\phi(x-y\cot\theta,y\csc\theta)}{p-1}\,d\theta \\
&\quad = \int \lim_{p\to 1}\frac{\phi(s+(1-p)(x-s),pr)-\phi(s,r)}{p-1}\,d\mu_{x,y}(s,r) \\
&\quad = \int\left((s-x)\frac{\partial\phi}{\partial s}+r\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r) \\
&\quad = T^*\left((s-x)\frac{\partial\phi}{\partial s}+r\frac{\partial\phi}{\partial r}\right)(x,y).
\end{aligned}
$$

LEMMA 3.4. *Assume that ϕ is a compactly supported smooth function; then*

$$\int\left(r^2\frac{\partial\phi}{\partial s}+r(s-x)\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r)=0\quad \textit{for all } x \textit{ and } y.$$

PROOF. Since ϕ is a smooth function, for all x and y

$$\lim_{\theta\to 0}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\eta_\theta\right)=\lim_{\theta\to 0}\phi(x-y\cot\theta,y\csc\theta)$$

and

$$\lim_{\theta\to \pi}\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\eta_\theta\right)=\lim_{\theta\to \pi}\phi(x-y\cot\theta,y\csc\theta)$$

are the same. Hence

$$
\begin{aligned}
0 &= \int_{\theta=0}^{\pi}\frac{\partial}{\partial\theta}(\phi(x-y\cot\theta,y\csc\theta))\,d\theta \\
&= \int_{\theta=0}^{\pi}\frac{\partial}{\partial s}\phi(x-y\cot\theta,y\csc\theta)\frac{\partial s}{\partial\theta} \\
&\quad + \frac{\partial}{\partial r}(x-y\cot\theta,y\csc\theta)\frac{\partial r}{\partial\theta}\,d\theta.
\end{aligned}
$$

But

$$\frac{\partial s}{\partial\theta}=\frac{d}{d\theta}(x-y\cot\theta)=y\csc^2\theta=\frac{r^2}{y},$$

and

$$\frac{\partial r}{\partial\theta}=\frac{d}{d\theta}(y\csc\theta)=-y\cot\theta\csc\theta=\frac{(s-x)r}{y}.$$

This gives

$$0=\int\left(\frac{r^2}{y}\frac{\partial\phi}{\partial s}+\frac{(s-x)r}{y}\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r),$$

hence

$$0=\int\left(r^2\frac{\partial\phi}{\partial s}+(s-x)r\frac{\partial\phi}{\partial r}\right)d\mu_{x,y}(s,r).$$

THEOREM 3.5. *For all compactly supported smooth functions ϕ and $\mathcal{P}$,*

$$(\Delta T^*\phi) = -T^*(\Box\phi).$$

PROOF. Using the decomposition of Δ given previously, we have

$$\Delta(T^*\phi) = \left[\left(y\frac{\partial}{\partial y}\right)^2 - y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial x^2}\right](T^*\phi).$$

By Lemmas 3.2 and 3.3, this is equal to

$$T^*\left[\left((s-x)\frac{\partial}{\partial s} + r\frac{\partial}{\partial r}\right)^2 - \left((s-x)\frac{\partial}{\partial s} + r\frac{\partial}{\partial r}\right) + y^2\left(\frac{\partial}{\partial s}\right)^2\right]\phi$$

$$= T^*\left[(s-x)^2\frac{\partial^2\phi}{\partial s^2} + 2(s-x)r\frac{\partial\phi}{\partial s\partial r} + y^2\frac{\partial^2\phi}{\partial s^2}\right].$$

Now by Lemma 3.4, the above is equal to

$$T^*\left[(s-x)^2\frac{\partial^2\phi}{\partial s^2} - 2r^2\frac{\partial^2\phi}{\partial s^2} + r^2\frac{\partial^2\phi}{\partial r^2}\right].$$

T^* is an integral over all geodesics (s, r) passing through the point $x + iy$, hence $(s-x)^2 + y^2 = r^2$ holds. Thus the preceding equation now becomes

$$\Delta T^* = T^*\left[r^2\frac{\partial^2\phi}{\partial s^2} - r^2\frac{\partial^2\phi}{\partial r^2}\right]$$
$$= -T^*(\Box\phi). \quad \text{Q.E.D.}$$

For the reverse property, $\Box(Tf) = -T(\Delta f)$, we shall state the lemmas and leave the proofs to the reader, but we will include an abbreviated proof of the theorem. Throughout, f is a compactly supported smooth function on $\mathcal{H} = \{x + iy \colon y > 0\}$.

LEMMA 3.6.

$$T\left(\frac{\partial f}{\partial x}\right)(s, r) = \frac{\partial}{\partial s}(Tf(s, r)).$$

LEMMA 3.7.

$$T\left((x-s)\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)(s, r) = r\frac{\partial}{\partial r}(Tf(s, r)).$$

LEMMA 3.8.

$$T\left((x-s)y\frac{\partial f}{\partial y} - y^2\frac{\partial f}{\partial x}\right) = 0.$$

THEOREM 3.9. $\Box(Tf) = -T(\Delta f)$.

PROOF.

$$\Box = r^2\frac{\partial^2}{\partial s^2} - r^2\frac{\partial^2}{\partial r^2} = r^2\frac{\partial^2}{\partial s^2} - r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + r\frac{\partial}{\partial r}.$$

Now by Lemmas 3.6 and 3.7 we have

$$\Box(Tf)(s, r) = T\left(r^2\frac{\partial^2 f}{\partial x^2} - (x-s)^2\frac{\partial^2 f}{\partial x^2} - 2(x-s)y\frac{\partial^2 f}{\partial x\partial y} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s, r).$$

But by Lemma 3.8,

$$T\left((x-s)y\frac{\partial^2 f}{\partial x \partial y}\right) = T\left(y^2\frac{\partial^2 f}{\partial x^2}\right).$$

Hence

$$\Box(Tf)(s,r) = T\left((r^2-(x-s)^2) - 2y^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s,r).$$

Now T is an integral over points (x,y) belonging to a geodesic (s,r), hence $(x-s)^2+y^2=r^2$ holds. Therefore

$$\begin{aligned}\Box Tf(s,r) &= T\left(-y^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}\right)(s,r)\\ &= -T(\Delta f)(s,r). \quad \text{Q.E.D.}\end{aligned}$$

4. Inversion formula. We begin this section with a discussion of horocycle transforms, studied extensively by Helgason. A more detailed version of the background information can be found in Helgason [**9**], Terras [**18**], and Vergne [**19**]. A horocycle can be defined as a trajectory which is orthogonal to a family of geodesics. For the upper half-plane $\mathcal{H}$, a horocycle is a circle tangent to the x-axis or a line parallel to the x-axis.

EXAMPLE.

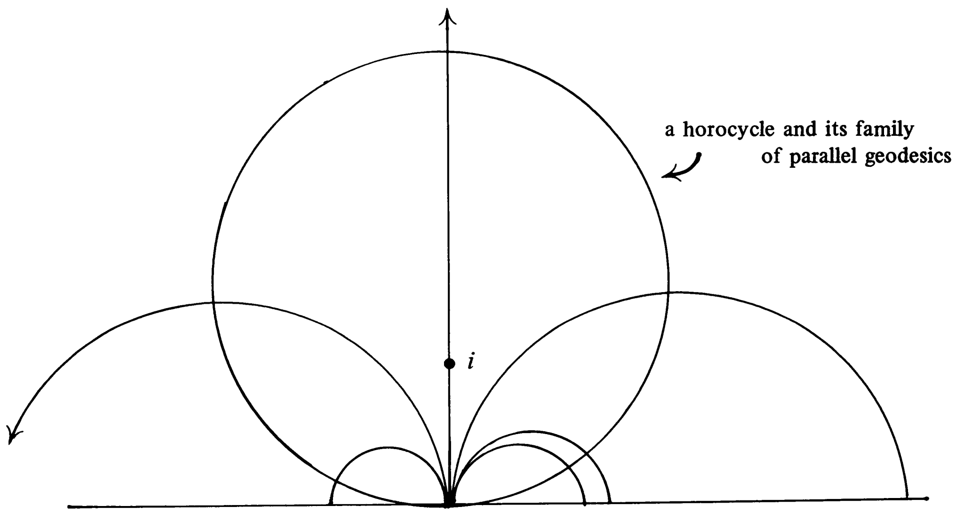

FIGURE 1

Horocycles are orbits $\{R(\theta)A(t)N(n)i\colon -\infty < n < +\infty\}$ in terms of the Iwasawa decomposition of G. The example shows a horocycle with $\theta = \pi/2$. The point of tangency, $b = -\cot\theta$, is called the "normal" of the horocycle. Define the bracket $\langle z,b\rangle$ to be the directed non-Euclidean distance from i to the horocycle passing through z and the boundary point b. Obviously, $\langle z,b\rangle = \langle w,b\rangle$ if z and w lie on the same horocycle with the boundary point b. Also, $\langle z,b\rangle$ is negative if the point i is inside the horocycle. To relate the X-ray transform in $\mathbf{R}^2$ with Fourier analysis, the horocycles (straight lines in $\mathbf{R}^2$) are parametrized by the directed distances from the origin $(0,0)$ and an angle θ. The quantity $\langle z, b\rangle$ in $\mathcal{H}$ is the natural analog of the directed distance in $\mathbf{R}^2$. The function defined by Helgason [**9**],

$$e_{\lambda,b}\colon z \mapsto e^{\lambda\langle z,b\rangle}, \qquad \lambda \in \mathbf{C},\ z \in \mathcal{H},$$

plays the role of the usual exponential function in what is called the *Helgason transform* for the symmetric space G/K. (See Terras [**18**].) The purpose of this section is to give a decomposition of the Helgason transform in terms of the Radon transform. We begin by stating the inversion formula and properties for $e_{\lambda,b}$ found in Helgason.

DEFINITION. For a compactly supported smooth function f on $\mathcal{H}$, define $\tilde{f}(\lambda, b)$ by

$$\tilde{f}(\lambda, b) = \int_{\mathcal{H}} f(z) e^{(1-i\lambda)\langle z,b\rangle}\, dz.$$

THEOREM 4.1 (HELGASON).

$$f(z) = \int_{\lambda\in\mathbf{R}} \int_{\theta=0}^{2\pi} \tilde{f}(\lambda, -\cot\theta) e^{(1-i\lambda)\langle z,b\rangle}\, d\mu,$$

where $d\mu(\lambda, b) = (1/4\pi^2)\tanh(\pi\lambda)\, d\lambda\, d\theta$.

PROPOSITION 4.2 (HELGASON). *Let g be an element of $G = SL(2,\mathbf{R})$, and $g\cdot z$, the Möbius transform of z by g; then*

$$\langle g\cdot z, g\cdot b\rangle = \langle z, b\rangle + \langle g\cdot i, g\cdot b\rangle.$$

PROPOSITION 4.3 (HELGASON). $|d(g\cdot b)/db| = e^{2\langle g^{-1}\cdot i, b\rangle}$.

The proofs of 4.1–4.3 are in Helgason [**9**].

PROPOSITION 4.4. *Let $z = x + iy$, $y > 0$. Let b be a boundary point in $\mathbf{R}$ of $\mathcal{H}$. Then*

$$\langle z, b\rangle = \ln\left[\frac{(1+b^2)y}{(x-b)^2+y^2}\right].$$

PROOF. $\langle z, b\rangle$ is the shortest distance from i to the horocycle determined by z and b. Since the Möbius transforms permute horocycles and preserve non-Euclidean distances, we select one which preserves the point i and moves the horocycle to a horizontal line (i.e., takes b to $\pm\infty$). A Möbius transformation which does this is

$$g = (1+b^2)^{-1/2}\begin{pmatrix} b & 1 \\ -1 & b \end{pmatrix}.$$

If we parametrize points on the original horocycle as $w = (b + r\cos\theta) + i(r + r\sin\theta)$, then it is easy to verify that the imaginary part of $g\cdot w$ is a constant for all θ. In fact, $\operatorname{Im}(g\cdot w) = (1+b^2)/2r$. In terms of $w = u + iv$, this is

$$\operatorname{Im}(g\cdot w) = \frac{(1+b^2)v}{(u-b)^2+v^2}.$$

Lastly, it is trivial to see that the non-Euclidean distance from i to the horizontal horocycle (above) is

$$\ln\frac{(1+b^2)y}{(x-b)^2+y^2}$$

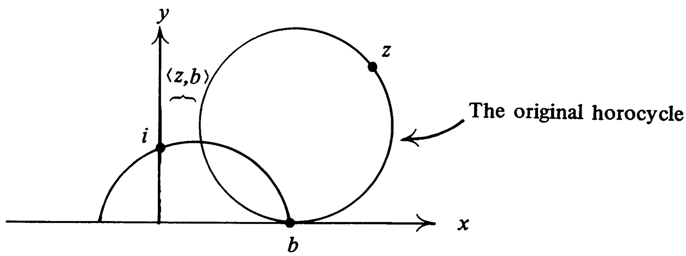

FIGURE 2

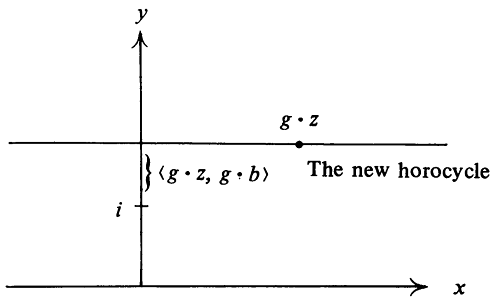

FIGURE 3

(see Figures 2 and 3). Note that this is just a special case of 4.2 because $\langle i, b\rangle = 0$ for all b.

REMARK. The horocycle containing z and having the normal b can be parametrized as $\{R(\theta)A(t)N(n)i\colon\ -\infty < n < +\infty\}$, where θ and t are chosen so that $-\cot\theta = b$ and $z = R(\theta)A(t)N(n_0)$ for some n_0. The Möbius transformation g in the proof is $R(-\theta)$. The distance from i to the horocycle is $\langle z, b\rangle = t$.

Let us now define the horocycle (Radon) transform, $Lf(\eta)$, for each compactly supported smooth function on $\mathcal{H}$ and a horocycle η by

$$Lf(\eta) = \int_\eta f(z)\, d\sigma_\eta(z)$$

where $d\sigma_\eta$ is the invariant measure on η. If we parametrize the space of horocycles by $\{R(\theta)A(t)N(n)i\colon\ -\infty < n < +\infty\}$, then $d\sigma$ on η is dn. Two horocycles $\eta_{\theta_1,t_1} = \{R(\theta_1)A(t_1)N(n)i\colon\ -\infty < n < \infty\}$ and $\eta_{\theta_2,t_2} = \{R(\theta_2)A(t_2)N(n)i\colon -\infty < n < \infty\}$ are the same if and only if either $\theta_1 = \theta_2$ and $t_1 = t_2$ or $\theta_1 - \theta_2 = \pi$ and $t_1 = -t_2$.

Another parametrization of the space of horocycles is (b, r) where b is the normal of the horocycle and r is the radius of the circle. Hence

$$\{(r\cos\alpha + b) + i(r\sin\alpha + r)\colon 0 \le \alpha < 2\pi\}$$

parametrizes the horocycle. Then $d\sigma$ is $d\alpha/(1 + \sin\alpha)$, and the integral becomes

$$Lf(b,r) = \int_{\alpha=0}^{2\pi} f(b + r\cos\alpha, r\sin\alpha + r)\frac{d\alpha}{1+\sin\alpha}.$$

It is now easy to see the following result.

THEOREM 4.5. *$\tilde{f}(\lambda, b) = ((1+b^2)/2)^{1-i}M_r[Lf(b,r)](i\lambda - 1)$ where M_r is the Mellin transform in the variable r.*

PROOF.

$$\begin{aligned}\tilde{f}(\lambda, b) &= \int f(x,y)e^{(-i\lambda+1)\langle x+iy,b\rangle}\frac{dxdy}{y^2}\\ &= \int f(x,y)\left(\frac{(1+b^2)y}{(x-b)^2+y^2}\right)^{1-i\lambda}\frac{dxdy}{y^2}.\end{aligned}$$

Write $x = r\cos\alpha + b$ and $y = r\sin\alpha + r$; then its Jacobian is $r + r\sin\alpha$. Here $dxdy/y^2 = (d\alpha/(1+\sin\alpha))(dr/r)$. Making the change of variables, we have

$$\begin{aligned}\tilde{f}(\lambda, b) &= \int_{r=0}^{\infty}\int_{\alpha=0}^{2\pi} f(r\cos\alpha + b, r\sin\alpha + b)\left(\frac{1+b^2}{2r}\right)^{1-i\lambda}\frac{d\alpha}{1+\sin\alpha}\frac{dr}{r}\\ &= \left(\frac{1+b^2}{2}\right)^{1-i\lambda}\int_{r=0}^{\infty} Lf(b,r)r^{i\lambda-1}\frac{dr}{r}.\end{aligned}$$

REMARK. By the support theorems in Helgason [**9**], L is one-to-one on $C_c^\infty(\mathcal{H})$. On the other hand $f \to \tilde{f}$ is also one-to-one on $C_c^\infty(\mathcal{H})$ and an inversion formula is known. See Terras [**18**]. Denote by $g \to \hat{g}$ the inverse of the Helgason transform $f \to \tilde{f}$. Write

$$M(Lf) = \left(\frac{1+b^2}{2}\right)^{1-i\lambda}\int_{r=0}^{\infty}(Lf)(b,r)r^{i\lambda-1}\frac{dr}{r};$$

then $\widehat{M(Lf)} = f$ by the above comments if $f \in C_c^\infty(\mathcal{H})$.

Before we proceed, we take another look at Theorem 4.5. In the theorem, the Helgason transform was written as a composition of the horocycle transform followed by the Mellin transform with a multiplicative factor. The multiplicative factor is not really necessary if we use the Iwasawa decomposition to parametrize the space of horocycles as will be seen below:

Let us fix an angle θ_0. Write every point z in $\mathcal{H}$ as $R(\theta_0)A(t)N(n)i$ for some t and n. Then $dz = dx\,dy/y^2 = dt\,dn$. A horocycle $\eta = \{R(\theta_0)A(t)N(n)i\colon -\infty < n < +\infty\}$ is normal to $b = -\cot\theta_0$ and its distance to the point i is t. Hence

$$\tilde{f}(\lambda, -\cot\theta_0) = \int f(R(\theta_0)A(t)N(n)i)e^{(1-\lambda_i)t}\,dt\,dn.$$

On the other hand, the integral

$$\int \tilde{f}(R(\theta_0)A(t)N(n)i)\,d\eta$$

is an integral over a horocycle η with the measure $d\sigma_\eta = d\eta$. Hence this is exactly the horocycle transform $Lf(\eta)$. We now have the following theorem.

THEOREM 4.6. *Let the space of horocycles be parametrized by $\{R(\theta)A(t)N(n)i\colon -\infty < n < \infty\}$ where $0 < \theta < \pi$ and $-\infty < t < \infty$. Then*

$$\tilde{f}(\lambda, -\cot\theta) = \int_{-\infty}^{\infty} Lf(\theta, t)e^{(1-\lambda_i)t}\,dt,$$

where $Lf(\theta,t)$ is the horocycle transform

$$Lf(\theta,t)=\int_{-\infty}^{+\infty} f(R(\theta)A(t)N(n)i)\,d\eta.$$

In order to do the analogous theorem for the X-ray transform (i.e., the transform over geodesics) it is necessary to invert a Mellin-Fourier type transform which does for geodesics what the Helgason transform does for horocycles. For the Helgason transform, one regards every point on the upper half-plane as belonging to a unique element of the set of horocycles with normal b. If we wish to to the same for geodesics, we can look at every point on the upper half-plane as belonging to a unique geodesic with "center" b. We note that Gel′fand, Gindikin, and Shapiro [**2**] define this to be an "admissible family" of curves.

Now we need a notion of distance, so define $\langle\langle z,b\rangle\rangle$ to be the non-Euclidean directed distance from $i+b$ to the geodesic containing z with center b. It is trivial to see that $\langle\langle z,b\rangle\rangle=\ln r$ for $z=x+iy$ with $(x-b)^2+y^2=r^2$. We define an exponential function $E_{\nu,b}$ by

$$E_{\nu,b}\colon z\mapsto \exp(\nu\langle\langle z,b\rangle\rangle).$$

So we have $E_{\nu,b}(z)=r^\nu$. We define the following transform:

$$\hat{f}(\nu,b)=\int_{\mathcal{H}} f(z)E_{\nu,b}(z)\,dz=\int f(z)((x-b)^2+y^2)^{\nu/2}\,\frac{dxdy}{y^2}.$$

If we assume f has compact support then $\hat{f}(\nu,b)$ must converge. We then have the following result.

THEOREM 4.7. *If f is a compactly supported smooth function and if $g(x,y)=f(x,y)y^{-1}$ then*

$$\hat{f}(\nu,b)=M_r[Tg(b,r)](\nu+1),$$

where T is the X-*ray transform over the geodesic with center b and radius r, and M_r is the Mellin transform in the variable r.*

PROOF.

$$\hat{f}(\nu,b)=\int_{\mathcal{H}} f(z)((x-b)^2+y^2)^{\nu/2}\,\frac{dxdy}{y^2}.$$

Letting $x=r\cos\alpha+b$ and $y=r\sin\alpha$ we have its Jacobian r, so

$$\begin{aligned}\hat{f}(\nu,b)&=\int f(r\cos\alpha+b,r\sin\alpha)r^\nu\frac{r\,dr\,d\alpha}{r^2\sin^2\alpha}\\&=\int g(r\cos\alpha+b,r\sin\alpha)r^{\nu+1}\frac{dr}{r}\frac{d\alpha}{\sin\alpha}.\end{aligned}$$

Switching the order of integration will give us the desired result.

In conclusion, we would like to point out that although the Radon transform for the upper half-plane has been known for a long time, it has not been used much as a number-theoretical tool. With sufficient development it should be possible to use it as such. L. Ehrenpreis also has some results for the horocycle case which make use of a point of view different from that of this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, FLORIDA 33199